Memory effects and macroscopic manifestation of randomness

A. A. Stanislavsky*
Institute of Radio Astronomy, 4 Chervonopraporna Street, 61002 Kharkov, Ukraine
(Received 1 October 1999)

It is shown that due to memory effects the complex behavior of components in a stochastic system can be transmitted to macroscopic evolution of the system as a whole. Within the Markov approximation widely used in ordinary statistical mechanics, memory effects are neglected. As a result, a time-scale separation between the macroscopic and the microscopic level of description exists, the macroscopic differential picture is not a consequence of microscopic nondifferentiable dynamics. On the other hand, the presence of complete memory in a system means that all its components have the same behavior. If the memory function has no characteristic time scales, the correct description of the macroscopic evolution of such systems has to be in terms of the fractional calculus.

PACS number(s): 05.40.−a, 05.45.−a, 05.60.−k

I. INTRODUCTION

The connection between microscopic dynamics of components in stochastic systems and macroscopic description of their behavior as a whole is very attractive in statistical physics [1–3]. From the point of view of a single trajectory, the path of a Brownian particle is continuous everywhere, but is nowhere differentiable [4]. Such a path is not described by an analytical function. In the theory of Brownian motion formulated by Langevin [5] the velocity of the Brownian particle was proved to be discontinuous [6]. The differentiable nature of the macroscopic picture of Brownian motion is due to the key role of the central limit theorem (fluctuations of the microscopic quantities are independent of each other). This means that the microscopic and macroscopic levels of description of the process are separated in the time scale, and memory of the nondifferentiable character peculiar to the microscopic dynamics is lost in the long time limit. Consequently, the results of observing the motion of an ensemble of trajectories can be predicted by means of theoretical prescriptions based on ordinary mathematical procedures proceeding from the differentiability assumption. When the condition of time-scale separation is not available, the nondifferentiable nature of the microscopic dynamics can be transmitted to the macroscopic level [7]. In the present paper we show that the key to this understanding is memory effects in stochastic systems. It is due to memory effects that the macroscopic behavior of stochatic systems contains a manifestation of microscopic dynamics.

The outline of the paper is as follows. Section II is devoted to the behavior of relaxation in the physical systems without and with complete memory. In Sec. III we use the generalized (in the terms of the memory effects) Langevin and Kramers-Moyal equations to show that, by means of the memory function, the nondifferentiable nature of microscopic dynamics of system components can be transmitted to the macroscopic level of description in the form of fractional derivatives. Note that the memory effects can induce orderedness of macroscopic processes in stochastic systems. In Sec. IV we briefly consider the criterion of the relative degree of order in the systems. Section V is devoted to the time fractional diffusion-wave equation concerning the case when chaos and order coexist. Finally we compare the process described by the latter equation with the fractional Brownian motion.

II. CLASSIFICATION OF MEMORY EFFECTS

We start from some classification of memory effects. It is based on mathematical properties of the corresponding memory function. Let us consider the integro-differential equation

$$\frac{df(t)}{dt} = -\lambda^2 \int_0^t K(t-t')\, f(t')\, dt', \tag{1}$$

where f is the quantity of interest, K the memory kernel, and λ the parameter. The equation is a typical non-Markovian equation obtained in studying the physical systems coupled to an environment, with environmental degrees of freedom being averaged. The parameter λ can be regarded as the strength of the perturbation induced by the environment of the system.

For a system without memory (ideal Markov system), the time dependence of the memory function $K(t-t')$ is of the form

$$K(t-t') = \delta(t-t'), \tag{2}$$

where $\delta(t-t')$ is the Dirac δ function. The absence of memory means that the convolution function, $J(t) = \int_0^t K(t-t')\, f(t')\, dt'$, is defined by $f(t)$ at the only instant t. Substituting Eq. (2) into Eq. (1) we obtain

$$\frac{df(t)}{dt} = -\lambda^2 f(t). \tag{3}$$

Equation (3) has an exponential solution. If memory effects are introduced into the system the δ function in Eq. (2) turns into a bell-shaped function, with the width determining an interval τ during which $f(t)$ has an effect on the function J.

*Electronic address: alexstan@ira.kharkov.ua

In the literature, wide use is made of the Markov approximation, which replaces Eq. (1) with the rate equation

$$\frac{df(t)}{dt} = -\left(\lambda^2 \int_0^\infty K(t')\,dt'\right) f(t). \tag{4}$$

The Van Hove limit [8] implies that the limit $\lambda \to 0$, $t \to \infty$ is approached in such a way that the product $\lambda^2 t$ is kept constant. Using the Van Hove limit makes it possible to replace the time convolution in Eq. (1) with

$$\frac{df(t)}{dt} = -\lambda^2 \tau f(t), \tag{5}$$

where

$$\tau = \int_0^\infty dt\, K(t).$$

The limit $\lambda \to 0$ implies that the coupling of the system to the environment is weak, while the limit $t \to \infty$ means that the observation time is much larger than the characteristic time scale τ.

On the other hand, in the systems having ideal complete memory the function J is formed over all the course of the action of the quantity $f(t')$ up to the instant t with the weight $K(t-t') = \{1, 0 < t' \leqslant t; 0, t' > t\}$ (step function). In this case Eq. (1) is transformed into

$$\frac{df(t)}{dt} = -\lambda^2 \int_0^t f(t')\,dt'. \tag{6}$$

It has the unique solution, $f(t) = f(0)\cos(\lambda t)$, which does not decrease at $t \to \infty$ in contrast to Eq. (3).

Relation (1) written in the time domain is not always convenient because of the convolution (integral over t'). This can be eliminated by using the Laplace transformation

$$\overline{f}(s) = \mathcal{L}[f(t)] = \int_0^\infty f(t)\, e^{-st}\, dt,$$

$$f(t) = \frac{1}{2\pi i} \int_{-i\infty}^{+i\infty} \overline{f}(s)\, e^{st}\, ds.$$

In this case Eq. (1) reduces to the algebraic form

$$s\,\overline{f}(s) - f(0) = -\lambda^2 \overline{K}(s)\,\overline{f}(s), \tag{7}$$

where the initial condition is taken into account. The Laplace transform of the kernel (2), which corresponds to the absence of memory, yields the constant, $\overline{K}(s) = 1$. For ideal memory, we obtain $\overline{K}(s) = 1/s$. Thus, as the ideal complete memory appears in the system the constant kernel is replaced by the hyperbolic one. It is logical to infer that the fractional integration of the order ν, $0 < \nu < 1$, will interpolate the memory function between the δ function and step function. Systems with such a memory function occupy an intermediate position between the two limiting cases and are described in [9]. They have complete but not ideal memory. This means that the memory manifests itself within all the interval $(0,t)$ but not at each instant t'. Such a memory function has no characteristic time scale, i.e., the Markov approximation is inapplicable.

Assume that memory holds only at the points of a Cantor set. The problem lies in finding the Laplace transform of a step memory function. To construct a Cantor set, we first choose the entire time interval of the length T and remove the central part of the interval leaving two intervals of the length ξT (where $\xi < 1/2$). Obviously, to avoid the loss of the integral memory, the heights of the two resulting bars must be increased to the value $(2\xi T)^{-1}$. In the next stage, each remaining interval of the length ξT is subjected to the same division process. In each subsequent stage n, this contraction procedure is performed for the 2^{n-1} stages obtained in the preceding stage. One can easily see that the memory function $K_n(t)$ is represented by a set of 2^n bars of the height $1/(2\xi)^n T$ and of the width $\xi^n T$. The Laplace image of $K_n(t)$ is written as

$$\overline{K}_n(s) = \frac{1 - \exp(-sT\xi^n)}{sT\xi^n} \prod_{k=0}^{n-1} \frac{1 + \exp(-z\xi^k)}{2},$$
$$z = (1-\xi)Ts. \tag{8}$$

For $n \gg 1$ we have $|sT\xi^n| \ll 1$. As the special investigations show [9] (Chap. 5), the limiting value of the function $\overline{K}_n(s)$, when the number of divisions generating the Cantor set tends to infinity, becomes

$$\overline{K}(s) = (sT)^{-\nu}\, q[\ln(sT)], \tag{9}$$

where $\nu = \ln 2/\ln(1/\xi)$ is the fractal dimension and $q[\ln(sT)]$ is the periodical function with the period $\ln \xi$. From the physical point of view the fractal dimension ν informs us about the relative amount of states being conserved in the process of interaction and represents a quantitative measure of memory effects. It is clear from the analysis of the limiting case. For an empty Cantor set ($\nu = 0$) the dependence (9) reduces to the constant, corresponding to the entire absence of memory (in the t domain it consists of two δ functions located on the edges of the Cantor set). The limiting value of the similarity parameter $\xi = 1/2$ yields the dimension $\nu = 1$, which corresponds to the complete memory. The result (9) is correct for any sT from the interval

$$1/(1-\xi) \leqslant |sT| < \infty$$

or for the time variable located in the interval

$$0 < t/T < 1.$$

Averaging the function $q(\ln(sT))$ in Eq. (9) over the period $\ln \xi$ (see the details in [9]) and taking the inverse Laplace transform of $\overline{J}(s)$, we obtain the temporal fractional integral

$$J(t) \simeq \frac{1}{\Gamma(\nu)} \int_0^t (t-t')^{\nu-1} f(t')\,dt',$$

where $\Gamma(\nu)$ is the Gamma function (see the Appendix). The averaging procedure converts the discrete fractal density ρ_n

$=1/(2\xi)^n T$ to its continuous value, $\rho(t)\propto t^{\nu-1}$. Now the kernel $K(t-t')$ in Eq. (1) becomes

$$K(t-t')\sim\frac{1}{\Gamma(\nu)}(t-t')^{\nu-1},$$

indicating the presence of the fractional integral. The integral representation of Eq. (1) is equivalent to a differential equation of the fractional order. The above result (9) can be generalized to the case of an arbitrary number j of elementary blocks participating in the construction of a Cantor set. It turns out [9] that the result (9) remains true given that $\nu=\ln j/\ln(1/\xi)$. The transition from the regular Cantor sets to the case, when the parameter ξ is random in each stage of constructing a Cantor set, leads to the same result.

Thus, the systems with residual memory have $\overline{K}(s)\simeq s^{-\nu}$, where the exponent value, $0\leqslant\nu\leqslant 1$, determines the extent of memory preservation. Substituting it into Eq. (7) (let all constants be contained in λ^2) and using the inverse Laplace transformation, the solution of Eq. (1) takes the form

$$f(t)=f(0)\,E_{\nu+1}(-\lambda^2 t^{\nu+1}),$$

where $E_\alpha(z)=\Sigma_{k=0}^{\infty}z^k/\Gamma(\alpha k+1)$ is the one-parameter Mittag-Leffler function [10]. In the particular cases $\nu=0$ and $\nu=1$ we have

$$E_1(-z)=e^{-z},\quad E_2(-z^2)=\cos z.$$

For $0<\nu<1$ the quantity $f(t)$ has an algebraic decay as $t\to\infty$. Therefore memory effects can essentially change the character of relaxation. The above constructions are not only pure theoretical, but reflect the experimental situation. In this connection it should be mentioned that, for example, the relaxation curves of the experiments [11] on glassy material with the remarkable accuracy show the algebraic decay rather than the standard exponential relaxation. The power-law relaxation can be expected to be a common feature of dynamical systems in a transition region between the stochastic and regular motion (supercooled liquids, glasses, and polymer materials) [12].

III. QUALITATIVE KINETIC ANALYSIS OF RANDOM PROCESSES WITH MEMORY

It is well known [13] that the generalized Langevin equation is of the form

$$\frac{dv}{dt}=-\gamma\int_0^t M(t-t')\,v(t')\,dt'+L(t),$$

where v is the particle's velocity, M some memory function, and L the noise term. If the value $v(t)$ is observed with a time resolution $\delta t\gg\tau_c$, where τ_c is the correlation time of forces producing the random particle motion, the Markov approximation is applicable, so that we obtain the ordinary Langevin equation

$$\frac{dv}{dt}=-\gamma v(t)+L(t).$$

In the case of complete but not ideal memory the generalized Langevin equation becomes a fractional differential equation. The evolution of the probability distribution associated with the velocity in the fractional Langevin equation is described by the fractional partial differential equation in the phase space for the phenomenon [14].

Consider the probability distribution $P(x,t)$ to find a particle at point x and time t. The normalization condition is

$$\int_{-\infty}^{\infty}P(x,t)\,dx=1. \tag{10}$$

A broad class of various stochastic processes is given by the Chapman-Kolmogorov equation,

$$\frac{\partial P(x,t)}{\partial t}=\int M(x,t;x',t')\,P(x',t')\,dx'(t'), \tag{11}$$

where the memory function $M(x,t;x',t')$ accounts for the probability distribution $P(x',t')$ in the previous instants of time $t'<t$ [15]. The expansion of the kernel M in terms of the difference $x-x'$ yields the generalized (in the terms of the memory effects) Kramers-Moyal equation

$$\frac{\partial P(x,t)}{\partial t}=\sum_{n=1}^{\infty}(-\nabla)^n\int_{-\infty}^{t}D^{(n)}(x,t-t')\,P(x,t')\,dt', \tag{12}$$

where $\nabla\equiv\partial/\partial x$ and the coefficients $D^{(n)}$ are the moments of the memory function M divided by $n!$. In the case of Markovian processes the moments are proportional to $\delta(t-t')$ and the integration in Eq. (12) vanishes. It is relevant to remark that t in Eq. (12) should be treated in the ''kinetic'' but not ''microscopic'' sense. The Markov approximation (the Van Hove method [8]) ignores memory effects in some sense, but such an approach is not always useful. For example, the statistical theory describing transport properties of turbulent plasma leads to the conclusion that the turbulence is of a subdiffusive nature and that the diffusivity considerably decreases. Therefore, memory effects can be important for explaining the dependence of the transport properties of saturated turbulence on the eigenfrequency of the unstable mode in the case of instability driven by the gradients in the coordinate space [16].

Using Eq. (12) one can write the equation

$$\frac{\partial P(x,t)}{\partial t}=\sum_{n=1}^{\infty}(-1)^n\frac{1}{n!}\frac{\partial^n}{\partial x^n}\left[K_n(x)\times\int_0^t D(t-t')\,P(x,t')\,dt'\right], \tag{13}$$

where $K_n(x)$ are arbitrary functions. The application of the Laplace transform to Eq. (13) [with the initial values given on the whole real axis in the form $P(x,0)$] leads to the following nonhomogeneous differential equation:

$$s\overline{P}(x,s)-P(x,0)=\overline{D}(s)\sum_{n=1}^{\infty}(-1)^n\frac{1}{n!}\frac{\partial^n}{\partial x^n}[K_n(x)\,\overline{P}(x,s)].$$

Consider the case when memory is complete but not ideal (see the previous paragraph). Then we come to the following equation:

$$s\overline{P}(x,s)-P(x,0)=\frac{D_1}{s^{\nu}}\sum_{n=1}^{\infty}(-1)^n\frac{1}{n!}\frac{\partial^n}{\partial x^n}[K_n(x)\,\overline{P}(x,s)], \tag{14}$$

where D_1 is the positive constant. Using the Caputo's definition (A6) of the fractional derivative (see [17] and the Appendix) and its Laplace transform (A8), the study of Eq. (13) for the case of complete but not ideal memory leads to the fractional generalization of the Kramers-Moyal equation:

$$\frac{\partial^{2\beta}P(x,t)}{\partial t^{2\beta}}=\sum_{n=1}^{\infty}(-1)^n\frac{1}{n!}\frac{\partial^n}{\partial x^n}[K_n(x)\,P(x,t)]. \tag{15}$$

If $K_1(x)$, $K_2(x)$ exist (nonzero) and $K_n(x)=0$ for $n\geqslant 3$, Eq. (15) is a fractional generalization of the Fokker-Planck-Kolmogorov equation [18].

Observe some typical features of the fractional Fokker-Planck-Kolmogorov (FFPK) equation. If $\beta=1/2$ ($\nu=0$) it is transformed into the conventional diffusion equation. This form is equivalent to the complete absence of memory. For $\beta=1$ ($\nu=1$) we have the conventional wave equation, i.e., the process with complete memory. The equations containing derivatives of higher than second order with respect to time cannot exist in nature: a random process cannot spread faster than a collection of deterministic trajectories. $\beta=0$ defines the case of localization, which is the lowest limit of any diffusion process. Hence the physical bounds on β are given by $0\leqslant\beta\leqslant 1$ [from Eq. (14) it follows that $1/2\leqslant\{\beta=(\nu+1)/2\}\leqslant 1$ because $0\leqslant\nu\leqslant 1$]. As is well known from statistical physics, one of the simple criteria of irreversibility is whether or not equations are invariant in respect to time reversal ($t\rightarrow -t$). The specific character of the processes described by fractional time derivatives is that for the substitution

$$(-t)^{2\beta}=t^{2\beta}\{\cos(2\beta\pi)+i\sin(2\beta\pi)\}$$

the relative amount of system states is conserved, and the other one corresponds to irreversible losses [9]. This allows us to suppose that for $0<\beta<1$ the FFPK equation describes the random processes with memory.

IV. CRITERION FOR THE RELATIVE DEGREE OF ORDER

The FFPK equation is an integro-differential equation in partial derivatives with varying coefficients, so in general one cannot analytically find its solution. For the stationary systems, the probability distribution does not depend on time $[(\partial/\partial t)P(x,t)=0]$, and their analysis leads to the Gibbs distribution, the cornerstone of statistical physics. A natural extension of the stationary analysis is the study of nonstationary systems in the self-similar regime, when the dependence on two arguments x,t is expressed in terms of a single variable $y=x/a(t)$:

$$P(x,t)=[a(t)]^{\alpha}\,\varphi(y), \tag{16}$$

where the functions $a(t),\varphi(y)$ and the exponent α are to be defined. Mathematically, the probability distribution (16) is a homogeneous function of order α. Physically, the transition to the new variable $y=x/a$ corresponds to scaling the stochastic quantity x on an arbitrary scale $a(t)$. It is well known [19] that such a feature is displayed by fractal objects. If the domain of definition of the phase space for the stochastic system is a fractal set, its dimension D lies between 2 (the conventional phase space) and 0 (the point of equilibrium).

In order to find the exponent α, we substitute the function (16) into the normalization condition (10), getting as a result

$$[a(t)]^{-(1+\alpha)}=\int_{-\infty}^{\infty}\varphi(y)\,dy.$$

The left-hand side of this equation depends on the time, whereas the right-hand side does not. Hence, it follows that $\alpha=-1$. The form of the function $\varphi(y)$ can only be found from the FFPK equation.

Of different macroscopic functions, only the entropy S possesses a combination of properties that allow one to use it as a measure of uncertainty (chaos) in the statistical description of the processes in macroscopic systems [3],

$$\begin{aligned}S(t)&=-\int_{-\infty}^{\infty}P(x,t)\log[P(x,t)]\,dx+S_0\\&=-\int_{-\infty}^{\infty}\varphi(y)\log[\varphi(y)]\,dy+\log[a(t)]+S_0.\end{aligned} \tag{17}$$

Define the mean value of x^2 as

$$\langle x^2\rangle=a^2(t)\int_{-\infty}^{\infty}y^2\,\varphi(y)\,dy=Ba^2(t).$$

Using the above relation we can write expression (17) in the form

$$\begin{aligned}S(t)=&-\int_{-\infty}^{\infty}\varphi(y)\log[\varphi(y)]\,dy-0.5\log B\\&+0.5\log\langle x^2\rangle+S_0.\end{aligned} \tag{18}$$

The entropy $S(t)$ depends on time only on account of $\langle x^2\rangle$. To consider the evolution of stationary states by means of slowly varying controlling parameters (they can be found among the parameters characterizing the stationary state) one can use the S theorem [20] as a criterion of the relative degree of order in various states which reveals for what states the degree of order is higher. It should be noted that the S-theorem considers only the stationary states using thermal equilibrium as the reference point for the degree of chaos. At $t\rightarrow\infty$ the probability distribution becomes negligible and it is necessary to renormalize the entropy,

$$\tilde{S}=S(t)-0.5\log\langle x^2\rangle. \tag{19}$$

The procedure is equivalent to fixing the value $\langle x^2\rangle$ for any value of a chosen controlling parameter (by the way, one of conditions of the S theorem is the equality condition for the

average effective Hamiltonian functions in different states of interest), since this influences only the reference point of entropy and does not affect the renormalized entropy differences between different states. Although the value $\langle x^2\rangle$ depends on the controlling parameter (as well as on t), the fixing of the former does not mean fixing the latter. Now using the renormalized entropy difference as a measure of the relative degree of order, one can study the evolution of system states in the space of controlling parameters.

V. SOME EXAMPLES

One of the simple cases of the FFPK equation is the time fractional diffusion-wave (TFDW) equation

$$\frac{\partial^{2\beta}}{\partial t^{2\beta}}P(x,t)=D\frac{\partial^2}{\partial x^2}P(x,t), \tag{20}$$

where D is a positive constant. Its fundamental solutions in case of the basic Cauchy and Signalling problems are well known [21]. Let us take the point $P(\xi,0)=\delta(\xi)$ as an initial position of a particle. If $1/2<\beta\leqslant 1$, it is necessary to specify the initial value of the first order time derivative $(\partial/\partial t)P(x,t)|_{t=0^+}$, since in this case two linearly independent solutions are to be determined. To ensure the continuous dependence of our solution on the parameter β in the transition from $2\beta=1^-$ to $2\beta=1^+$, we assume $(\partial/\partial t)P(x,t)|_{t=0^+}=0$. Then Eq. (20) has the following solution:

$$P(x,t)=\frac{1}{2\sqrt{D}t^{\beta}}M\left(\frac{|x|}{\sqrt{D}t^{\beta}};\beta\right), \tag{21}$$

where

$$M(z;\beta)=\frac{1}{2\pi i}\int_{Ha}e^{\sigma-z\sigma^{\beta}}\frac{d\sigma}{\sigma^{1-\beta}},\quad 0<\beta<1$$

where Ha denotes the Hankel path [a contour that begins at $\sigma=-\infty-ia$ $(a>0)$, encircles the branch cut that lies along the negative real axis, and ends up at $\sigma=-\infty+ib$ $(b>0)$].

Mainardi [21] developed the function $M(z;\beta)$ as a series

$$M(z;\beta)=\sum_{n=0}^{\infty}\frac{(-1)^n z^n}{n!\,\Gamma[-\beta n+(1-\beta)]}$$

and showed that it is a particular case of the Wright function

$$W(z;\lambda,\mu)=\sum_{n=0}^{\infty}\frac{z^n}{n!\,\Gamma(\lambda n+\mu)}=\frac{1}{2\pi i}\int_{Ha}e^{\sigma+z\sigma^{\lambda}}\frac{d\sigma}{\sigma^{\mu}},$$

where $\lambda>-1$, $\mu>0$ [10]. It is non-negative for any $0<\beta\leqslant 1$ and satisfies the normalization condition $\int_0^{\infty}M(\zeta;\beta)\,d\zeta=1$. Clearly, these properties are also characteristic to the probability distribution mentioned above. For $\beta=1$ the function is the Dirac δ function. For $\beta=1/2$ we have the Gaussian function $M(z;1/2)=\exp(-z^2/4)/\sqrt{\pi}$. For $0<\beta\leqslant 1/2$ the function $M(z;\beta)$ $(z\geqslant 0)$ decreases monotonically, while for $1/2<\beta<1$ it first increases and then decreases, exhibiting its maximum value at a certain point z_{max}. The expression (21) describes the particle evolution in the space-time and is the Green function of the TFDW equation. Averaging x^2 we obtain

$$\langle x^2\rangle=\int_{-\infty}^{\infty}x^2\,P(x,t)\,d\xi=\frac{2\,Dt^{2\beta}}{\Gamma(2\beta+1)}. \tag{22}$$

The probability distribution (21) obeys the following scaling relations:

$$P(\hat{x}=b^{\beta}x,\hat{t}=bt)=b^{-\beta}P(x,t)$$

for an arbitrary parameter b.

Note that if in (21) $\beta=1/2$ we obtain the conventional Brownian motion (purely random process). For $\beta=1$ the particle performs ballistic motion (purely deterministic process). Next we are going to show that $0<\beta<1$ is the special case when chaos and ordered motion coexist.

Now let us calculate the entropy (18) in this case

$$S(t)=0.5+0.5\log_2(\langle x^2\rangle)+S(\beta). \tag{23}$$

As a controling parameter we choose β and will consider the evolution of the sequence of states corresponding to different values of the controlling parameter. One should keep in mind that $\lim_{t\to\infty}\langle\xi^2\rangle=\infty$. For this reason we must renormalize the expression (23) according to Eq. (19). Then the relative degree of order can be estimated as

$$S(\beta)=0.5\log_2\Gamma(2\beta+1)-\int_0^{\infty}M(z,\beta)\log_2[M(z,\beta)]\,dz. \tag{24}$$

The dependence $S(\beta)$ is represented in Fig. 1. It reaches the maximum at $\beta_c=1/2$. We regard this state as a state of physical chaos (the correctness of this assumption will have to be verified). The smooth decrease of the value (24) is a quantitative measure of the increase in the degree of order. Since the inequality $S(1/2)>S(\beta\neq 1/2)$ is satisfied, $\beta\to\frac{1}{2}+\Delta\beta$ is the transition from a less ordered state (physical chaos) to a more ordered state. This is an indication that we have found the corresponding controlling parameter, and the evolution of the system in the space of the controlling parameter is associated with self-organization. The conclusion is valid because for $\beta=1$ we have the purely deterministic state which may be taken by a reference point of the degree of order. Some orderedness also happens to be the case for $0<\beta<1/2$. Although here we do not come to complete order, in some sense the orderedness is higher than for $\beta_c=1/2$. Thus the value β can be adopted as a measure of relative degree of order. It is useful to mention some particular values, $\lim_{\beta\to 0}S(\beta)=1/\ln 2$ and $S(1/2)=0.5(1+\ln\pi)/\ln 2$. To sum up, the process considered above is the special case when chaos and ordered motion coexist, the value β characterizing the relative degree of order in the process. Next we will compare this process with the fractional Brownian motion.

In many physical systems the Gaussian distribution is a straightforward consequence of the central limit theorem which makes it possible to consider completely random processes. Hurst found a set of statistical tools to examine the data which does not represent a purely random structure

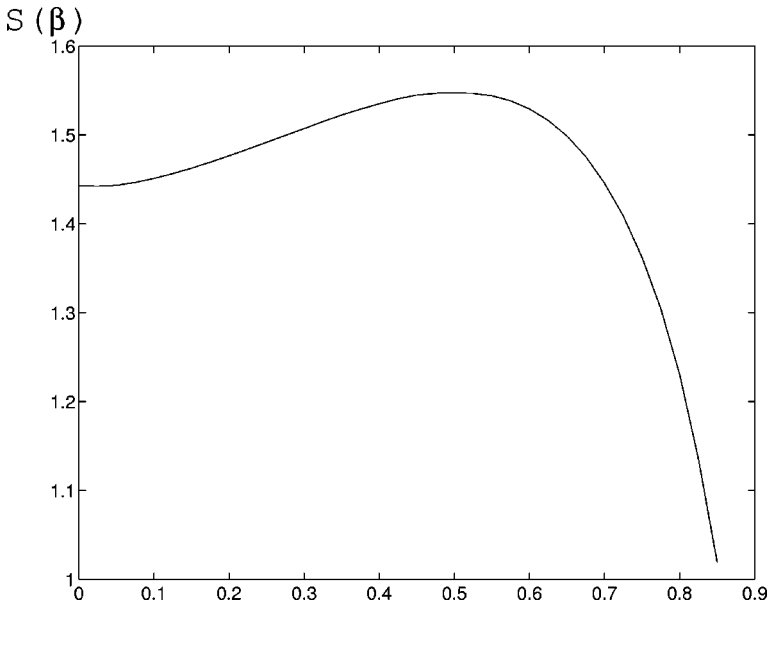

FIG. 1. The dependence of the value $S(\beta)$ on the controlling parameter β.

though standard statistical methods do not show any correlation between the observations [22]. Using his rescaled range analysis one can extract meaningful information about the ''memory'' of a time series. If the observations are not independent, each carries a ''memory'' of events which preceded it. Mandelbrot and Van Ness [23] suggested the so-called fractional Brownian motion (the fractional integral of the white Gaussian noise) as a model reflecting the phenomenon. The fractional Brownian motion has the following probability distribution:

$$\Pi(x;t)=\frac{1}{\sqrt{2\pi K}\,t^{\beta}}\exp\left(-\frac{x^2}{4K\,t^{2\beta}}\right). \tag{25}$$

Its average $\langle x^2\rangle$ is proportional to $t^{2\beta}$ as in the case of the process (21). Then expression (18) is of the form

$$S(t)=0.5\{\ln[2\pi\hat{D}(t)]+1\}, \tag{26}$$

where $\hat{D}(t)=2K\,t^{2\beta}$. Renormalizing it in accordance with Eq. (19) we obtain $\tilde{S}=0.5(\ln 2\pi+1)$. In other words, the renormalized entropy does not depend on β, i.e., the fractional Brownian motions with various Hurst exponents are almost the same in respect to the relative degree of chaos. This example differs radically from the foregoing one: in the model of fractional Brownian motion (25) the rise of orderedness with changing the Hurst exponent is impossible for the simple reason that the shape of the probability distribution does not change in the space of the controling parameter [compare with the process (21)]. The numerical modeling verifies this conclusion [24].

VI. CONCLUDING REMARKS

The fractional calculus formalism generalizing differentiation and integration to fractional orders has a long history [25], but recently the interpretative approach opened broad perspectives in physical and engineering applications [12,14,18]. The above consideration shows that fractional calculus provides a bridge between purely deterministic processes and purely stochastic ones. The fact is of interest in its own right because chaos and order in Nature coexist. According to Boltzmann and Gibbs, in closed statistical systems, evolution in time results in the equilibrium state which is the most chaotic (or, in other words, purely random). The element motions in closed systems are independent of each other. In open systems the environment induces memory effects, so that the macroscopic behavior of such systems contains a manifestation of microscopic dynamics. If microscopic motion in the systems is very complex or random, the complete memory transmits the complex behavior to macroscopic evolution of the system as a whole. It is worth noticing that the model (used by us) of memory at the points of a Cantor set is not too exotic. It implies that memory is intrinsic to all time scales (in such a way that the corresponding memory function has no characteristic scales) of the phase space of a system given that the number of divisions generating a Cantor set tends to infinity. In a different way, the largest of time scales would be in the system as in the case of the Markov approximation. We have demonstrated that the relationship between the Cantor set and the fractional integral reduces the generalized (in the terms of the memory effects) Langevin, Kramers-Moyal, Fokker-Planck-Kolmogorov equations to their fractional form. The fractional generalizations turn out to be useful for studying the random processes with residual memory (without any characteristic time scale).

The process described by the time fractional diffusion-wave equation is an example clearly showing that for chaos and ordered motion in a system to coexist, its probability distribution must undergo qualitative changes in the function form with slowly varying the system parameters. In our consideration the parameter was the relative amount of the system states having orderedness during the system evolution. The evolution in the space of the parameter looks like a stochastic analog of bifurcation connected with the phase transition ''order-disorder.'' For $0<\beta<1/2$ the probability distribution has one sharp maximum at $z_{max}=0$. If the pa-

rameter attains the value $\beta=1/2$ the function becomes flat. Then for $1/2<\beta<1$ the probability function takes the form with two maxima symmetrical in respect to the origin of coordinates (where the function minimum is found). When the parameter β goes towards 1, the probability to find a particle between the maxima is becoming less and lesser, and the function peaks get narrower and higher. For $\beta=1$ the probability distribution is transformed in two δ functions. As a result, this system state becomes completely ordered. The fractional Brownian motion has not the possibility. The Hurst exponent (which is the only parameter of the process) influences the asymtotic behavior of its autocorrelation function, but the process remains Gaussian anyhow.

ACKNOWLEDGMENTS

I appreciate helpful discussions with Professor Raoul Nigmatullin (Jordan State University) and thank Professor Francesco Mainardi (University of Bologna) for useful comments.

APPENDIX

In this appendix we consider briefly the basic formulas used in the fractional calculus introduced in the text. The starting point is the introduction of the causal function $\Phi_\lambda(t)$ defined as

$$\Phi_\lambda(t) := \begin{cases} \dfrac{t^{\lambda-1}}{\Gamma(\lambda)}, & t \geqslant 0, \\ 0, & t<0, \end{cases} \qquad \lambda \in \mathbf{C},$$

whose Laplace transform is

$$\mathcal{L}[\Phi_\lambda(t)] = \overline{\Phi}_\lambda(s) := \int_0^{+\infty} e^{-st}\,\Phi_\lambda(t)\,dt = \frac{1}{s^\lambda}, \qquad \mathrm{Re}\,\lambda>0, \quad \mathrm{Re}\,s>0. \tag{A1}$$

The function satisfies the composition rule

$$\Phi_\lambda(t)*\Phi_\mu(t) = \int_0^t \Phi_\lambda(\tau)\,\Phi_\mu(t-\tau)\,d\tau = \Phi_{\lambda+\mu}(t), \qquad \mathrm{Re}\,\lambda>0, \quad \mathrm{Re}\,\mu>0.$$

The integral of order n of a causal function $f(t)$ can be expressed by the convolution between Φ_n and f

$$\begin{aligned} I^n f(t) &= \int_0^t dt_1 \int_0^{t_1} dt_2 \cdots \int_0^{t_{n-2}} dt_{n-1} \int_0^{t_{n-1}} f(\tau)\,d\tau \\ &= \frac{1}{(n-1)!}\int_0^t f(\tau)\,(t-\tau)^{n-1}\,d\tau \\ &= \Phi_n(t)*f(t), \end{aligned}$$

basing on the well known formula that reduces the calculation of the n-fold primitive to a single integral. For $\lambda>0$, the function $\Phi_\lambda(t)$ is locally absolutely integrable in $0\leqslant t<+\infty$. To extend the above formula from positive integer values of the index to any positive real values, let us define the fractional integral of order $\alpha>0$,

$$I^\alpha f(t) := \frac{1}{\Gamma(\alpha)}\int_0^t (t-\tau)^{\alpha-1} f(\tau)\,d\tau = \Phi_\alpha(t)*f(t). \tag{A2}$$

The Laplace transform of the fractional integral is the straigthforward generalization of the ordinary case $\alpha=n$

$$\mathcal{L}[I^\alpha f(t)] = \frac{\overline{f}(s)}{s^\alpha}. \tag{A3}$$

For $\lambda\leqslant 0$, the causal function $\Phi_\lambda(t)$ is not locally absolutely integrable and consequently the integral

$$\frac{1}{\Gamma(-\alpha)}\int_0^t \frac{f(\tau)}{(t-\tau)^{1+\alpha}}\,d\tau = \Phi_{-\alpha}(t)*f(t), \quad \alpha\in\mathbf{R}^+ \tag{A4}$$

is in general divergent. Nevertheless, if $\lambda=-n$ $(n=0,1,\ldots)$, the functions $\Phi_\lambda(t)$ can be treated in the framework of generalized functions [26]. In this case it reduces to the n derivative (in the generalized sense) of the Dirac δ function

$$\Phi_{-n}(t) := \delta^{(n)}(t).$$

Then formally the derivative of order n of a causal function $f(t)$ can be obtained by the convolution between $\Phi_{-n}(t)$ and f

$$\frac{d^n}{dt^n} f(t) := \int_0^t f(\tau)\,\delta^{(n)}(t-\tau)\,d\tau = \Phi_{-n}(t)*f(t), \quad t>0.$$

The limit case $\alpha=0$ defines the identity operator

$$\int_0^t f(\tau)\,\delta(t-\tau)\,d\tau = \Phi_0(t)*f(t) = f(t).$$

In order to obtain a definition for the fractional derivative (with noninteger α) that is valid for classical functions, we have to regularize the divergent integral (A4) in some way. As a consequence we arrive at two alternative regular definitions for the fractional derivative D^α, which read for $n-1<\alpha<n$

$$D_c^\alpha f(t) = \frac{1}{\Gamma(n-\alpha)}\,\frac{d^n}{dt^n}\int_0^t \frac{f(\tau)}{(t-\tau)^{\alpha+1-n}}\,d\tau, \tag{A5}$$

$$D_r^\alpha f(t) = \frac{d^\alpha}{dt^\alpha} f(t) = \frac{1}{\Gamma(n-\alpha)}\int_0^t \frac{f^{(n)}(\tau)}{(t-\tau)^{\alpha+1-n}}\,d\tau. \tag{A6}$$

The difference between the two definitions lies in

$$D_c^\alpha f(t) = D_r^\alpha f(t) + \sum_{k=0}^{n-1} f^{(k)}(0^+)\,\Phi_{(k-\alpha+1)}(t).$$

The defintion (A5) is the most commonly adopted in mathematically oriented papers and books [25]. The definition (A6), introduced by Caputo [17], is more restrictive than Eq. (A5) because it requires the function $f(t)$ to be n differentiable. As it was shown in this paper, the latter definitions are more suitable for the problems considered above where the conventional initial conditions are expressed in terms of integer derivatives. Using the classical technique of Laplace transform

$$\mathcal{L}\left[\frac{d^n}{dx^n}f(t)\right]=s^n\overline{f}(s)-\sum_{k=0}^{n-1}s^{n-1-k}f^{(k)}(0^+),\quad n\in\mathbf{N} \tag{A7}$$

and from Eqs. (A1) and (A6) we get

$$\mathcal{L}\left[\frac{d^\alpha}{dt^\alpha}f(t)\right]=s^\alpha\overline{f}(s)-\sum_{k=0}^{n-1}s^{\alpha-1-k}f^{(k)}(0^+),\qquad n-1<\alpha<n, \tag{A8}$$

first stated by Caputo [17]. It is worth noticing that according to the definition (A5) the fractional derivative of a constant does not vanish if α is not an integer, while according to defintion (A6) it vanishes for any α.

[1] G. M. Zaslavsky, Chaos **5**, 653 (1995); G. Gallovatti, *ibid.* **8**, 384 (1998).

[2] N. G. van Kampen, *Stochastic Processes in Physics and Chemistry* (North-Holland, Amsterdam, 1983); I. Prigogin and I. Stengers, *Order out of Chaos* (Heinemann, London, 1984).

[3] Yu. L. Klimontovich, Teor. Mat. Fiz. **96**, 385 (1993); Chaos, Solitons and Fractals **5**, 1985 (1995); Usp. Fiz. Nauk **166**, 1231 (1996) [Phys. Usp. **39**, 1169 (1996)].

[4] J. Perrin, Ann. Chim. Phys. Sept. (1909), translated by F. Soddy, *Brownian Movement and Molecular Reality* (Taylor and Francis, London, 1909).

[5] P. Langevin, C.R. Acad. Sci. (Paris) **530** (1908).

[6] J. L. Doob, Ann. Math. **43**, 351 (1942).

[7] B. J. West, *Physiology, Promiscuity and Prophecy at the Millennium: A Tale of Tails*, Studies of Nonlinear Phenomena in the Life Sciences Vol. 9 (World Scientific, Singapore, 1998); P. Grigolini, A. Rocco, B. J. West, Phys. Rev. E **59**, 2603 (1999); A. Rocco and B. J. West, Physica A **265**, 535 (1999).

[8] L. Van Hove, Physica (Amsterdam) **21**, 517 (1955); R. W. Zwanzig, in *Lectures in Theoretical Physics*, Boulder, 1960 (Interscience Publishers, New York, 1961), Vol. III, p. 106.

[9] R. R. Nigmatullin, Theor. Math. Phys. (English edition) **90**, 245 (1992); Alain Le Mehaute, Raoul R. Nigmatullin, and Laurent Nivanen, *Fleches du Temps et Geometrie Fractale* (Hermes, Paris, 1998).

[10] *Higher Transcendental Functions*, edited by A. Erdélyi (McGraw-Hill, New York, 1955), Vol. 3, Chap. 18.

[11] W. G. Glöckle and T. F. Nonnenmacher, Macromolecules **24**, 6426 (1991).

[12] J. F. Douglas and J. B. Hubbard, Macromolecules **24**, 3163 (1991); J. F. Douglas, Comput. Mater. Sci. **4**, 292 (1995); J. Phys.: Condens. Matter **11**, A329 (1999).

[13] H. Mori, Prog. Theor. Phys. **33**, 423 (1965); **34**, 399 (1965).

[14] B. J. West and P. Grigolini, in *Applications of Fractional Calculus in Physics*, edited by R. Hilfer (World Scientific, Singapore, 2000).

[15] H. Risken, *The Fokker-Planck Equation* (Springer-Verlag, Berlin, 1989).

[16] A. Zagorodny and J. Weiland, Ukrain. J. Phys. **43**, 1402 (1998).

[17] M. Caputo, *Elasticitá e Dissipazione* (Zanichelli, Bologna, 1969).

[18] M. F. Shlesinger, G. M. Zaslavsky, and J. Klafter, Nature (London) **363**, 31 (1993); G. M. Zaslavsky, Physica D **76**, 110 (1994); H. E. Roman, Fractals **5**, 379 (1997).

[19] B. B. Mandelbrot, *The Fractal Geometry of Nature* (W.H. Freeman, New York, 1982).

[20] Yu. L. Klimontovich, Pis'ma Zh. Éksp. Teor. Fiz. **9**, 1409 (1983); Z. Phys. B: Condens. Matter **66**, 125 (1987).

[21] W. R. Schneider and W. Wyss, J. Math. Phys. **30**, 134 (1989); F. Mainardi, Izv. Vyssh. Uchebn. Zaved., Radiofiz. **38**, 20 (1995); Chaos, Solitons and Fractals **7**, 1461 (1996).

[22] H. E. Hurst, R. P. Black, and Y. M. Simaika, *Long-Term Storage: An Experimental Study* (Constable, London, 1965).

[23] B. B. Mandelbrot and J. W. Van Ness, SIAM (Soc. Ind. Appl. Math.) Rev. **10**, 422 (1968).

[24] J. Feder, *Fractals* (Pergamon, New York, 1988).

[25] K. B. Oldham and J. Spanies, *Fractional Calculus* (Academic Press, New York, 1974); S. G. Samko, A. A. Kilbas, and O. I. Marichev, *Fractional Integrals and Derivatives: Theory and Applications* (Gordon and Breach, Amsterdam, 1993).

[26] I. M. Gel'fand and G. E. Shilov, *Generalized Functions* (Academic Press, New York, 1964), Vol. 1.